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Analytical study of black tide model of QSOs

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Abstract : An analytical study of the quasar models employing a black hole in a galactic nucleus has been made The black hole is considered to be embedded in two-component isothermal spheres which grows (mainly by tidally disrupting stars) in galactic nucleus, and destroys the stellar population giving rise to the power source of QSOs, Seyfert nucleus, and radio galaxy phenomena Some important physical parameters, such as, the gravitational potential $\psi(\xi)$ and its derivative $\psi'(\xi)$ have been calculated. We have also given the values of ratio of density to central density $\varsigma(\xi)/\varsigma(0)$ and $\Phi(\xi)$ for two chosen values of $\mu(= m_2/m_1$ = ratio of 'heavy' particles to 'light' particles), equal to 2 and 3 Runs of ratio of total mass of heavy particles to the whole mass M_2/M and the energy T with radial distance ξ have been illustrated More importantly, the closed form expressions for the mass apportionment M_2/M_1 which might cause destabilizing effects in the system and be responsible for heavy particles to sink to the centre displacing the lighter particles outward, have been obtained. The simple and closed form expression for the consumption rate $\Phi(\xi)$ of the central black hole has been derived In addition, the general scenario of rate of fall of mass onto the star has been presented.

Keywords : Black hole, quasar models, galactic nucleus

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1. Introduction

Several authors [1–4] have explained the quasar activity as a closely associated phenomena with galactic nuclei. Our studies suggest that some galaxies (for example, normal galaxies) have weak radio sources and others have compact. The presence of a black hole (a huge mass of the order of 10^8 times the solar mass confined in a small volume) or high stellar density in galactic nuclei could be the reason for such activity (including emissions of gravitational, infrared and ultraviolet radiations of Quasistellar objects (QSO)s and Seyfert [5,6]). In other words, involvement of quantities of matter comparable to the galactic nucleus could be held responsible for above kinds of activity and a power output of $10^{11}\ L_{\odot}$ over a period of 10^7 yr. [7]. Hill's studies [7] support the viewpoint that the tidal breakup of stars could be responsible for above. His works also include various types of associated physical phenomena, such as, capture of gas (produced by the tidal breakup) by the black hole, luminosity and total energy of accreted material and rate of growth and luminosity of the black hole. More importantly, he pointed out that if mass M of the black hole is less than $3 \times 10^8 M_\odot$ then the tidal radius would be greater than the gravitational radius and the stars are far more likely to be broken apart by the black hole than the absorbed hole.

According to 'loss-cone' theory [8,9], the fluxes are not high enough as estimated by Hill since the consumption of stars depends on the diffusion rate of stellar orbits. Works of Peebles [10] and Bahcall and Wolf [11] support the theory of tidal disruption for causing the hole. Other authors, for example, Vaucher and Weedman [12] and Nolthenius and Katz [13] include in their discussions, various causes, such as, spin of the black hole causing power output, effects of tidal disruption processes and collisions of stars in the density cusp around the black hole.

Our theoretical studies have shown that the relaxed central regions of the highly evolved astronomical systems or massive objects, such as, black holes or supermassive stars residing at centres of star clusters and galactic nuclei (cores of rich clusters of galaxies) takes on a structure which closely resembles an isothermal sphere with small deviations from equilibrium (due only to escape of high velocity of stars) [14–16]. The observational evidence for globular cluster cores [17] favours these viewpoints. These considerations apply to large N-body stellar systems [18,19]. Hookey *et al* [20] and Masson [21] have described a net test of the redshift-angular-diameter-relation and have applied it to a sample of 3 CR and 4 C quasars. Marshall *et al* [22] have considered the phenomena of expanding quasars to obtain upper limit of internal proper motion and redshift. Detailed dynamical calculations of He'non [23] and of an isothermal core in such systems [24–27] pointed out the similarities between the density distributions of isothermal spheres and large-scale structure of clusters of galaxies. Turner *et al* [28] and Dyer [29] used angular isothermal model to study various properties of the lensing objects and their distribution in space.

In view of above, it seems appropriate here to introduce the theory of two-component isothermal spheres [30,31] for describing the galactic nucleus. Our method employs an approximate analytic analysis of the basic model of a black hole embedded in two-component isothermal spheres. As pointed out, since we are considering N-body stellar systems, the present treatment could be more advantageous (as would be more clear below) than one-component isothermal case [32] mainly from the viewpoint of mass apportionment, $M_2(\xi)/M(\xi)$ considerations which might cause destabilizing effects. The purpose of this paper is as follows : To obtain closed form (analytic) expressions for (i) the integrand $k(\varepsilon)$ appearing in the expression [11] for J_D (standard deviation of the angular momentum), (ii) consumption rate of stars by the central black hole, (iii) $M_2(\xi)/M(\xi)$ (ratio of total mass of heavy particles and total mass), (iv) $\varsigma(0)/\varsigma(\xi)$ ratio of central density to density at a point inside the model) and (v) T (dimensionless total energy).

Table 1 presents numerical values of some physical parameters $\psi(\varsigma)$ and $\psi'(\xi)$, and in Table 2 are given values of $\varsigma(0)/\varsigma(\xi)$ for $\mu(=m_2/m_1)$ equal to 2 and 3. The graphical picture shows runs of $M_2(\xi)/M(\xi)$ (Figure 1) and T (Figure 2) with radial

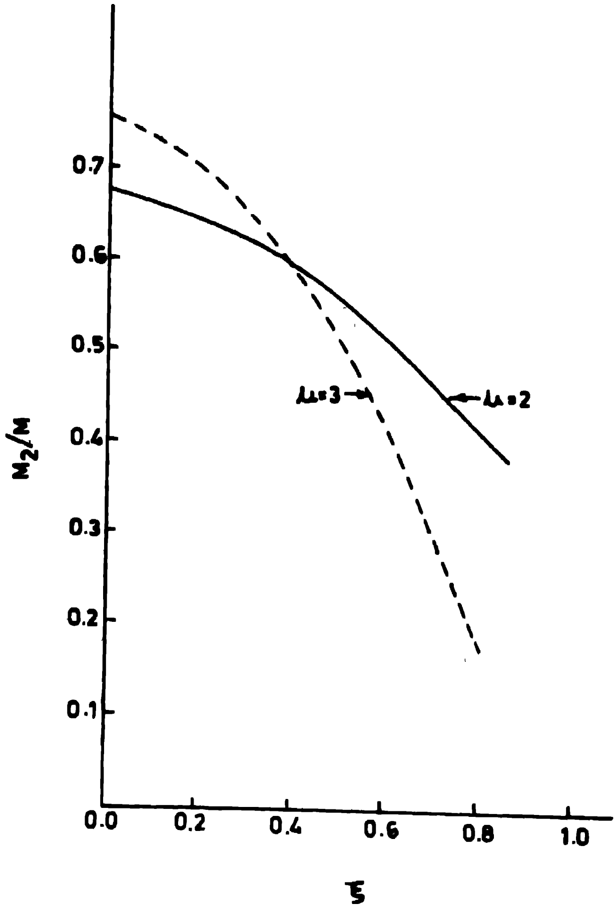

Figure 1. Run of the ratio of total mass M_2 of Heavy particles to the total mass M for μ (= m_2/m_1) = 2 and 3 (dashed curve). (ξ, M_2/M_1) curves are shown for the purpose of comparison.

distance ξ. We note, the mass apportionment cause destabilizing effects and may become a driving mechanism for sinking of heavier particles to the centre and displacing the lighter ones outward. The general scenario of rate of fall of mass onto the star is also presented.

2. Equilibrium equation for two-component isothermal spheres and their approximate analytical solutions

(i) Structure equations :

The dimensionless form of the generalized Emden equation governing the equilibrium of two-component isothermal, self-gravitating gas sphere, is given by

$$\frac{1}{\xi^N}\frac{d}{d\xi}\left(\xi^N \frac{d\psi}{d\xi}\right) = e^{-\psi} + \delta e^{-\mu\psi};$$

$$\psi(0) = \psi'(0) = 0 \quad (\psi' \equiv d/d\xi), \tag{1}$$

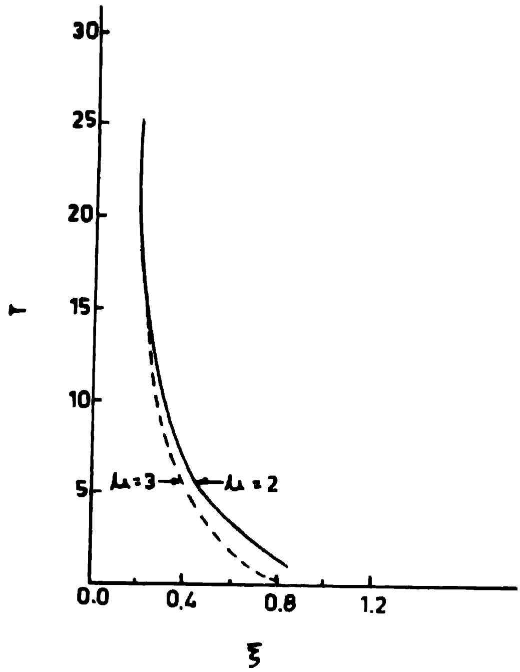

Figure 2. The total energy T Vs the dimensionless radial distance ξ for μ ($= m_2/m_1$) = 2 (solid curve) and 3 (dashed curve) is shown for the purpose of comparison

where the dimensionless radius ξ and gravitational potential $\psi(\xi)$ related to r and $\phi(\xi)$, respectively, are given by

$$\xi \equiv \left(4\pi G \varsigma_{01} m_1/k_B T\right)^{\frac{1}{2}} r \equiv \alpha r, \qquad \psi(\xi) \equiv m_1 \phi/kT, \tag{2}$$

where G, m_1, ς_{01} and T denote the gravitational constant, mass of lighter particles, central density and temperature, respectively; $\mu = m_2/m_1$, $\delta = \lambda u = \varsigma_{02}/\varsigma_{01}$, $= n_2/n_1$, and $N = 2$, 1 and 0 represent spherical, cylindrical and plane-symmetric configurations (suffixes '1' and '2' would mean the values for 'light' and 'heavy' particles of which the gas sphere is composed of).

The density profile $\varsigma_i(r)$ interior to r, and central density ς_{0i} are related by

$$\varsigma_i(r) = \varsigma_{0i} \exp\left[-m_i \phi(r)/k_B T\right] \tag{3}$$

which, in dimensionless form, can be expressed as

$$\varsigma(\xi) = m_1 n_1(0)\left[\exp(-\psi) + \lambda\mu \exp(-\mu\psi)\right], \tag{4}$$

or,

$$\varsigma(\xi) = \varsigma(0)(1+\lambda\mu)^{-1}\left[e^{-\psi} + \lambda\mu e^{-\mu\psi}\right]. \tag{4'}$$

3. Diffusion in two-component isothermal sphere

Approximate analytical solution of the diffused two-component isothermal sphere obtained previously by the author [33–35] is expressed as

$$\psi(\xi) = \frac{1+\lambda\mu}{2(N+1)}\xi^2\frac{1+A'\xi^2+B'\xi^4}{1+C'\xi^2+D'\xi^4}, \tag{5}$$

where A', B', C' and D' are constants which depend on chosen values of λ and μ. The structural length α (eq. (2)) can be re-expressed in the form

$$r \equiv \alpha^2\xi = \sigma_v^2\xi/4\pi G\varsigma_1(0); \qquad \sigma_v \equiv \left[k_BT/m_1\right]^{\frac{1}{2}}, \tag{6}$$

σ_v^2 denotes the square of one-dimensional dispersion velocity of the stars. The expressions for the ratio of density interior to the central density for small and large values of ξ can be written as

$$\varsigma(\xi)/\varsigma(0) = 1-(1+\lambda\mu^2)\frac{\xi^2}{6} \qquad (\xi \to 0), \tag{7}$$

$$\varsigma(\xi)/\varsigma(0) = \frac{1}{1+\lambda\mu}\left[p(\xi)+q(\xi)\right] \qquad (\xi \to \infty), \tag{8}$$

where
$$p(\xi) = \frac{2}{\xi^2}\left[1+\frac{A}{\xi^{1/2}}\cos\left(\frac{\sqrt{7}}{2}\log\xi-\delta\right)+\frac{\lambda\mu}{(2\mu^2-5\mu+4)}\left(\frac{2}{\xi^2}\right)^{\mu-1}\right],$$

$$q(\xi) = \left(\frac{2}{\xi^2}\right)^{\mu}\left[1+\mu\left\{\frac{A}{\xi^{1/2}}\cos\left(\frac{\sqrt{7}}{2}\ln\xi-\delta\right)\left(\frac{2}{\xi^2}\right)^{\mu-1}\right\}\right].$$

Table 1 presents numerical values of $\psi(\xi)$ and $\psi'(\xi)$:

Table 1. Numerical values of $\psi(\xi)$ and $\psi'(\xi)$.

μ \ ξ	2 ψ	3 ψ	2 ψ'	3 ψ'
0.00	0.000	0.000	0.000	0.000
0.20	0.020	0.026	0.196	0.198
0.40	0.077	0.099	0.361	0.385
0.60	0.165	0.205	0.459	0.519
0.80	0.276	0.330	0.428	0.483
1.000	0.392	0.464	0.100	0.157

The consumption rate Φ of the central black hole as obtained by Hills [7] can be written in the form

$$\Phi = \pi J_T^2 < v^{-1} > n_1(0) = J_T^2(2\pi)^{\frac{1}{2}}\sigma_v^{-1}n_1(0) \tag{9}$$

provided $J_D/J_T \geq 1$ (J_D denotes the root-mean-square diffusion in angular momentum). The diffusion equation in two-component isothermal sphere is given by

$$J_D^2(E) = 8\pi G^2 m_* \ln\wedge \int_0^{r_0} \varsigma(r) r^2 dr (\Phi - G)(x) \; / x^2 \sigma_\upsilon^2(r), \tag{10}$$

or with the help of eqs. (2) and (4) the foregoing equation can be written as

$$J_D^2(E) = 8\pi G^2 m_* \ln \Lambda \alpha^{-3} \varsigma_1(0) \sigma_\upsilon^{-2} \int_0^{\xi_0} \xi^2 d\xi \left[\exp(-\psi) + \lambda\mu \exp(-\psi\mu)\right]$$

$$\times \; (\Phi - G)(x)/x^2 = 2Gm_* \alpha^{-1} \ln\wedge K(\varepsilon), \tag{11}$$

where $k(\varepsilon)$ denotes the integrand in the foregoing equation, and $\ln \Lambda$ = 'Coulomb logarithm', $x^2 = \varepsilon - \psi(\xi)$, $\varepsilon = E/\sigma_\upsilon^2$, $\psi(\xi^*) = \varepsilon$. Significant progress can be made for obtaining suitable expressions for the consumption rate Φ in two limiting cases, $\varepsilon \ll 1$ and $\varepsilon \gg 1$, as given below :

Case I. $\varepsilon << 1$:

Near the centre ($\xi \to 0$) the integrand in eq. (11) can be re-expressed in the form

$$K(\varepsilon) = \int_0^{\xi_0} \xi^2 d\xi \left[1 + \lambda\mu\right] (\Phi - G)(x)/x^2, \tag{12}$$

where $\quad \psi(\xi) = (1 + \lambda\mu)\dfrac{\xi^2}{6}, \quad \varepsilon = \psi(\xi^*) \simeq (1 + \lambda\mu)\dfrac{\xi^{*2}}{6},$

$$\exp[-\psi(\xi)] = \exp[-\mu\psi(\xi)] = 1, \quad x^2 = (1 + \lambda\mu)\; \frac{\xi^{*2}}{6} - \frac{\xi^2}{6} \tag{13}$$

Further, we may approximate $(\Phi - G)(x) = \left(4/(3\sqrt{\pi})\right)x \; (x \to 0)$ then

$$K(\varepsilon) \simeq (1 + \lambda\mu)^{\frac{1}{2}} \left(4(2/3\pi)^{\frac{1}{2}}\right) \int_0^{\xi^*} \frac{\xi^2 d\xi}{\left[\xi^{*2} - \xi^2\right]^{1/2}} \simeq 2(1 + \lambda\mu)^{\frac{1}{2}} (6\pi)^{\frac{1}{2}} \varepsilon. \tag{14}$$

Case II. $\varepsilon >> 1$:

Since, for $\xi \to \infty$ the two-component isothermal shpere corresponds to the one-component isothermal case, we have

$$K(\varepsilon) \simeq \int_0^{\xi^*} 2d\xi \, (\Phi - G)(x)/x^2, \tag{15}$$

where $\quad x^2 = \varepsilon - \psi(\xi); \quad \varepsilon = E\sigma_\gamma^{-2}; \quad \psi(\xi^*) = \varepsilon,$

$$-\psi(\xi) \underset{\xi \to \infty}{\to} \ln\frac{2}{\xi^2} + \frac{A}{\xi^{1/2}} \cos\left(\frac{\sqrt{7}}{2} \ln \xi - \delta\right) + \frac{\lambda\mu}{(2\mu^2 - 5\mu + 4)} \left(\frac{2}{\xi^2}\right)$$

Finally, $k(\varepsilon)$ can be expressed as

$$k(\varepsilon) = 2\left[4/\left(4/(3\sqrt{\pi})\right)\right]\int_0^{\xi^*}\frac{d\xi}{\sqrt{\frac{2}{\xi^{*2}}-\frac{2}{\xi^2}}} \simeq (8/3)e^{-\varepsilon/2}erf\{(\varepsilon/2)\}. \tag{16}$$

We may express the flux Φ (into the loss cone) as sum of the two fluxes $\Phi_1(\varepsilon < \varepsilon_{\rm crit})$ plus $\Phi_2(\varepsilon > \varepsilon_{\rm crit})$; $\varepsilon_{\rm crit} = -\frac{E_{\rm crit}}{\sigma_v^2}$ that is,

$$\Phi = \Phi_1 + \Phi_2. \tag{17}$$

The expressions for Φ_1 and Φ_2 can be written as

$$\Phi_2 = 4\pi^2 J_T^2\int_{\varepsilon_{\rm crit}}^{\infty} dEf(E) = \Phi(-\varepsilon_{\rm crit}), \tag{18}$$

and
$$\Phi_1 = 2\pi^2\int_{\infty}^{\varepsilon_{\rm crit}} dEf(E)\left[J_D^2(\varepsilon)/\ln\wedge^*\right]. \tag{19}$$

Thus,
$$\Phi_1 = \Phi G m_* \alpha^{-1} J_T^{-2}\ln\wedge/\ln\wedge^*\int_0^{\varepsilon_{\rm crit}} d\varepsilon e^{-\varepsilon}k(\varepsilon). \tag{20}$$

In the limit $\varepsilon_{\rm crit} \leq 1$, we may find that $\Phi_1 + \Phi_2 \simeq \Phi$, and when $\varepsilon_{\rm crit} \geq 1$,

$$\int_0^{\varepsilon_{\rm crit}} d\varepsilon e^{-\varepsilon}k(\varepsilon) = \frac{8\sqrt{2}}{3} \quad \left(\varepsilon_{\rm crit} \to \infty\right) \tag{21}$$

$$\Phi_2 \to 0; \quad \Phi_1 = \Phi G m_* \alpha^{-1} J_T^{-2}\frac{8\sqrt{2}}{3}\ln\wedge/\ln\wedge^*, \tag{22}$$

And using eq. (9), we obtain

$$\Phi_\infty = \left(\frac{16\sqrt{x}}{3}\right)\left(G m_* \alpha^{-1} n_1(0)/\sigma_v\right)\left(\ln\wedge/\ln\wedge^*\right) \quad (\text{star s}^{-1}). \tag{23}$$

Table 2. Values of $\varsigma(0)/\varsigma(\xi)$ and $\Phi(\xi)$.

ξ	$\varsigma(0)/\varsigma(\xi)$		$\Phi(\xi)$	
	$\mu=2$	$\mu=3$	$\mu=2$	$\mu=3$
0.00	1.000	1.000	0.000 σ_v^2	0.000 σ_v^2
0.20	1.345	1.067	0.020 σ_v^2	0.026 σ_v^2
0.40	1.682	1.276	0.077 σ_v^2	0.099 σ_v^2
0.60	1.916	1.642	0.165 σ_v^2	0.205 σ_v^2
0.80	2.247	2.184	0.276 σ_v^2	0.330 σ_v^2
1.00	2.647	2.074	0.392 σ_v^2	0.464 σ_v^2

One may calculate the rate at which the stars collide in an isothermal sphere. In Table 2 are given values of $\varsigma(0)/\varsigma(\xi)$ and $\Phi(\xi)$ for $\mu(= m_2/m_1)$ equal to 2 and 3.

In this context, it would be interesting to examine the related question of mass apportionment which may produce destabilizing effects in the isothermal gas system [30,31]. This can be understood by the fact that heavy particles are drawn towards the centre while the lighter ones are displaced outward. The magnitude of the specific heat is increased and decreased, respectively, in the inner and outer regions. We note, the formation of binaries in the dense stellar system could produce destabilization not by soaking up large amount of negative energy and expelling single stars but by putting some of the mass into heavy particles [33]. The effects of mass approtionment can be studied as follows :

For small $\xi(\xi < \xi_1 = \mu)$, the ratio of total mass of heavy particles to the total mass $M(\xi)$ is given by

$$M_2(\xi) \simeq \frac{\delta}{1+\delta}\left[1 - \frac{\mu(1+\delta)\xi^2}{10}\right]. \tag{24}$$

The ratio of total density contrast can be written as

$$\varsigma(0)/\varsigma(\xi) = \frac{1+\delta}{\left[e^{-\psi} + \delta e^{-\mu\psi}\right]}. \tag{25}$$

The total energy T in dimensionless form can be expressed as [30]

$$T = -\frac{E}{GM^2/r} = -\frac{\chi}{\psi'(\xi)^2}\frac{(3/2)\left[1 - (M_2/M_1)(\mu-1)/\mu\right]}{\xi\psi'(\xi)}, \tag{26}$$

$$\chi = \exp(-\psi(\xi)) + \delta\mu^{-1}\exp(-\mu\psi(\xi)), \quad \psi'(\xi) \equiv \frac{d\psi}{d\xi}.$$

Plots of $M_2(\xi)/M(\xi)$ and T, for the two chosen values of $\mu = 2$ and $\mu = 3$ are shown in Figures 1 and 2, respectively. The energy being positive is indicative of the fact that (as in one-component isothermal case [26]) two-component polytropes may not satisfy the virial theorem. Moreover, there is scope for finding the value of $M_2(\xi)/M(\xi)$ for which $-rE/GM^2$ may become negative. Thus, it presents a brief idea about destabilizing effects of mass apportionment.

Apart from limitations on the distance $R_T\,(6M/\pi\varsigma)^{1/2}$ or R_G [7,34–36] :

$$R_G \simeq \frac{2GM}{c^2} \simeq \frac{2\sigma}{\alpha}U(\xi_1) \tag{27}$$

(where $U(\xi)$ and σ, respectively, denote the mass function and the relativistic parameter [7,36]) under which a star is either broken apart by the tidal forces or absorbed by the black hole. It could be interesting, in general, to cast a glance on the scenario of rate of fall of mass on the star :

Let there be a large number of particles each with mass m and speed v_∞ far from a star; and let their number density be n_∞. Then in Newtonian approximation, let $v_\infty << v_p = (2GM/R)^2$. ($v_p$ = parabolic velocity at the star's surface). The maximum angular momentum permitting capture is $l_{max} \simeq m v_p R$. To determine the impact parameter l_{max}, corresponding to infall tangentially onto the star's surface, we use the fact that $l_{max} = m v_\infty l$, hence $l_{max} = R(v_p/v_\infty)$. The flux of particles with $1 < l_{max}$ is given by

$$J_n v_\infty \pi (l_{max})^2. \tag{28}$$

Therefore, the rate of fall of mass ($\varsigma_\infty = mn_\infty$) onto the star can be expressed as

$$\left.\begin{aligned} dM/dt &= mv_\infty n\pi R^2 v_\infty^{-2}(2GM/R) \\ &= 2mv_\infty n\pi R^2 v_\infty^{-2} - c^2 \frac{U'(\xi_1)}{\xi_1} \end{aligned}\right\} \tag{29}$$

where R_G is given by (27). The critical *GTR* value for the angular momentum in accretion onto a frozen star is 2 mcR which means a mass-capture rate of

$$dM/dt = 4\pi R_G^2 c\varsigma_\infty (c/v_\infty). \tag{30}$$

Thus, from eqs. (29) and (30) it is clear that one cannot use Newtonian formula if $R/R_G < R$. The foregoing expression gives a lower limit on dM/dt and describes accretion not only onto frozen stars but also onto very dense neutron stars—those with $4 > R/R_G > 1.7$.

4. Conclusions

1. An analytical study of the quasar models (embedded in two-component isothermal sphere) which provide power source of QSOs, Seyfert nucleus, and radio galaxy phenomena has been made. The basic model is of a black hole.

2. Some important physical parameters, such as the gravitational potential $\psi(\xi)$, its derivative $\psi'(\xi)$, ratio of density to central density $\varsigma(\xi)/\varsigma(0)$, total mass of heavy particles to the whole mass M_2/M and energy T for two selected values of $\mu = 2$ and 3 (μ denotes ratio of 'heavy' particles to 'light' particles) have been presented in tabular and graphical form.

3. Destabilizing effects of mass apportionment M_2/M_1 which might provide strong constraints on the physical system have been noted.

4. Simple and closed form expressions for the consumption rate of the central black hole has been obtained. And in the last, cosmological phenomena of rate of fall of mass onto the star is briefly discussed.

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